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ENTROPY SOLUTIONS FOR A CLASS OF THE NONLOCAL (p, q)-LAPLACIAN PROBLEMS

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Abstract. In this paper, we consider the existence of entropy solutions to an elliptic (p, q)-Laplacian problem with homogeneous Dirichlet boundary condition when p and q are non-local quantities. We get the results by assuming the right-hand side function f to be an integrable function. The main goal of this paper is to extend the results established by M. Chipot and H.B. de Oliveira (Math. Ann., 2019, 375, 283-306).

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1 Introduction

We study the existence of entropy solutions for some variable exponent problems with exponents p, q that may depend on the unknown solution u. We consider the case where the dependency of p, q on u is a nonlocal quantity. Namely, we consider nonlocal Dirichlet boundary value problem of the following form

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(b(u))-2}\nabla u) - \mathrm{div}(|\nabla u|^{q(b(u))-2}\nabla u) = f \text{ in } \Omega \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω be a bounded domain of $\mathbb{R}^N, N \geq 2$, f is a given data, $p, q : \mathbb{R} \to [1, +\infty)$ are a real functions and $b : W_0^{1,\alpha}(\Omega) \to \mathbb{R}$.

By $W_0^{1,\alpha}(\Omega)$, we mean the Dirichlet-Sobolev space with constant exponent α satisfying $1 < \alpha < +\infty$ (that is, $W_0^{1,\alpha}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,\alpha}(\Omega)$). To underline the degree of generality in defining exponents p, q, we recall two typical examples of maps b of the following form:

$$b(u) = \|\nabla u\|_{L^\alpha(\Omega)}, \quad b(u) = \|u\|_{L^s(\Omega)}, s \leq \alpha^*,$$

namely, we may link $b(\cdot)$ to two norm definitions that are relevant from a mathematical point of view. Here, α^* denotes the critical Sobolev exponent of α.

In recent years, the existence, uniqueness, and regularity of solutions to the $(p(x), q(x))$-Laplacian problem have been studied in many works (Xiang et al. (2020); Yanru (2021); Zhang et al. (2019)). The situation where the variable exponents p, q depend on the unknown solution u is non-standard as in the classical case (see Abbassi et al. (2019, 2021); Akdim et al. (2019)). This kind of problems appear in the applications of some numerical techniques for the total variation image restoration method that have been used in some restoration problems of mathematical image processing and computer vision Blomgren et al. (1997); Bollt et al. (2007); Türola (2017). Türola (2017) have presented several numerical examples suggesting that the consideration of exponents $p = p(u)$ preserves the edges and reduces the noise of the restored images u. A numerical example suggesting a reduction of noise in the restored images u when the exponent of the regularization term is $p = p(|\nabla u|)$ is presented in Blomgren et al. (1997). Chipot et al. (2019) have proved the existence of weak solutions for some $p(u)$-Laplacian problems, the existence proofs of Chipot et al. (2019) are based on the Schauder fixed-point theorem. Andreianov et al. (2010), have studied the following prototype problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(u)-2}\nabla u) + u = f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Zhang et al. (2021) have proved the existence of entropy solutions for some nonlocal p-Laplacian type problems and they have provided some positive answers for the two questions proposed by Chipot and de Oliveira in Chipot et al. (2019). Ouaro et al. (2020) considered the following nonlinear Fourier boundary value problem

$$\begin{cases} b(u) - \mathrm{div}\, a(x, u, \nabla u) = f & \text{in } \Omega \\ a(x, u, \nabla u) \cdot \eta + \lambda u = g & \text{on } \partial\Omega. \end{cases}$$

The existence and uniqueness results of entropy solutions are established by an approximation method and convergent sequences in terms of Young measure. Yanru (2021) have obtained the existence of weak solutions of the $(p(u), q(u))$-Laplacian problem (1), where $(p(u), q(u))$ is a local quantity by means of singular perturbation technique and Schauder fixed point theorem.

The fact that in reality physical measurements of certain quantities are not made in a punctual way but through a local averages is always the motivation to study non-local problems. The main difficulty in the analysis of these $p(u)$-problems relies in the fact that their weak formulations cannot be written as equalities in terms of duality in fixed Banach spaces. The sequences of solutions u_n to these problems correspond to different exponents p_n and therefore belong to possible distinct Sobolev spaces.

This paper is organized us follow. In Sec. 2 we introduce the basic assumptions and we recall some definitions, basic properties of generalised Sobolev spaces that we will used later. The Sec. 3 is devoted to show the existence of entropy solutions to the local problem (1).

2 Preliminaries

In this section we introduce our notation and collect some useful materials. We focus on the setting of Lebesgue and Sobolev spaces with variable exponents, but we also link these spaces to their counterparts with constant exponents.

Let Ω be a bounded domain of $\mathbb{R}^N$ with $\partial\Omega$ Lipschitz-continuous, we say that a real-valued continuous function $p(\cdot)$ is log-Hölder continuous in Ω if

$$\exists C > 0: \ |p(x) - p(y)| \leq \frac{C}{\ln\left(\frac{1}{|x-y|}\right)} \quad \forall x, y \in \Omega, \quad |x - y| < \frac{1}{2}. \tag{2}$$

For any Lebesgue-measurable function $p : \Omega \to [1, \ \infty)$, we define

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \ , \ p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x), \tag{3}$$

and we introduce the variable exponent Lebesgue space by:

$$L^{p(\cdot)}(\Omega) = \{ \ u : \Omega \to \mathbb{R} \ / \ \rho_{p(\cdot)}(u) \ := \int_\Omega |u(x)|^{p(x)} dx < \infty \}. \tag{4}$$

Equipped with the Luxembourg norm

$$\|u\|_{p(\cdot)} := \inf\left\{\lambda > 0 \ : \ \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}, \tag{5}$$

$L^{p(\cdot)}(\Omega)$ becomes a Banach space. If

$$1 < p_- \leq p_+ < \infty, \tag{6}$$

$L^{p(\cdot)}(\Omega)$ is separable and reflexive. The dual space of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$, where $p'(x)$ is the generalised Hölder conjugate of $p(x)$,

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

The next proposition shows that there is a gap between the modular and the norm in $L^{p(\cdot)}(\Omega)$.

Proposition 1. *If* (6) *holds, for $u \in L^{p(x)}(\Omega)$, then the following assertions hold*

$$\min\left\{\|u\|_{p(\cdot)}^{p_-},\ \|u\|_{p(\cdot)}^{p_+}\right\} \leq \rho_{p(\cdot)}(u) \leq \max\left\{\|u\|_{p(\cdot)}^{p_-},\ \|u\|_{p(\cdot)}^{p_+}\right\},$$

$$\min\left\{\rho_{p(\cdot)}(u)^{\frac{1}{p_-}}\ ,\ \rho_{p(\cdot)}(u)^{\frac{1}{p_+}}\right\} \leq \|u\|_{p(\cdot)} \leq \max\left\{\rho_{p(\cdot)}(u)^{\frac{1}{p_-}}\ ,\ \rho_{p(\cdot)}(u)^{\frac{1}{p_+}}\right\}, \tag{7}$$

$$\|u\|_{p(\cdot)}^{p_-} - 1 \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p_+} + 1. \tag{8}$$

Proposition 2. *(Generalised Hölder's inequality)*
- For any functions $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have:

$$\int_\Omega uvdx \leq (\frac{1}{p_-} + \frac{1}{p'_-})\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

- For all p satisfying to (6)*, we have the following continuous embedding,*

$$L^{p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \ \textit{whenever}\ p(x) \geq r(x) \ \textit{for a.e.}\ x \in \Omega. \tag{9}$$

In generalised Lebesgue spaces, there holds a version of Young's inequality,

$$|uv| \leq \delta\frac{|u|^{p(x)}}{p(x)} + C(\delta)\frac{|v|^{p'(x)}}{p(x)},$$

for some positive constant $C(\delta)$ and any $\delta > 0$.
We define also the generalized Sobolev space by

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) : \nabla u \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space with the norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}. \tag{10}$$

The space $W^{1,p(\cdot)}(\Omega)$ is separable and is reflexive when (6) is satisfied. We also have

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega) \text{ whenever } p(x) \geq r(x) \text{ for a.e. } x \in \Omega. \tag{11}$$

Now, we introduce the following function space

$$W_0^{1,p(\cdot)}(\Omega) := \{u \in \mathrm{W}_0^{1,1}(\Omega) : \nabla u \in L^{p(\cdot)}(\Omega)\},$$

endowed with the following norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} := \|u\|_1 + \|\nabla u\|_{p(\cdot)}. \tag{12}$$

If $p \in C(\overline{\Omega})$, then the norm in $W_0^{1,p(\cdot)}(\Omega)$ is equivalent to $\|\nabla u\|_{p(\cdot)}$. When p is log-Hölder continuous, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$.
If p is a measurable function in Ω satisfying $1 \leq p_- \leq p_+ < N$ and the Log-Hölder continuity property (2), then

$$\|u\|_{p^*(\cdot)} \leq C\|\nabla u\|_{p(\cdot)} \quad \forall u \in W_0^{1,p(\cdot)}(\Omega),$$

for some positive constant C, where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

On the other hand, if p satisfies (2) and $p_- > N$, then

$$\|u\|_\infty \leq C\|\nabla u\|_{p(\cdot)} \quad \forall u \in W_0^{1,p(\cdot)}(\Omega)\ ,$$

where C is another positive constant.
Since the problem (1) is considered with integrable data, then it is reasonable to work with entropy solutions or renormalized solutions, which need less regularity than the usual weak solutions. We introduce the following definition of the truncation function T_k at height $k \geqslant 0$:

$$T_k(r) = \min\{k, \max\{r, -k\}\} = \begin{cases} k & \text{if } r \geqslant k \\ r & \text{if } |r| < k, \\ -k & \text{if } r \leqslant -k. \end{cases}$$

Next we define the very weak gradient of a measurable function u with $T_k(u) \in W_0^{1,p(b(u))}(\Omega)$. The proof follows from Lemma 2.1 of Benilan et al. (1995) due to the fact that $W_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,\alpha}(\Omega)$.

Proposition 3. *For every measurable function u with $T_k(u) \in W_0^{1,p(b(u))}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$, which we call the very weak gradient of u and denote $v = \nabla u$, such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \quad \text{for a.e. } x \in \Omega \text{ and for every } k > 0,$$

where χ_E denotes the characteristic function of a measurable set E.
Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v coincides with the weak gradient of u.

Lemma 1. *Chipot et al. (2019) Assume that*

$$1 < \alpha \leq q_n(x) \leq \beta < \infty \quad \forall n \in \mathbb{N},$$

$$\text{for a.e. } x \in \Omega, \text{ for some constants } \alpha \text{ and } \beta, \tag{13}$$

$$q_n \to q \ \text{ a.e. in } \Omega, \text{ as } n \to \infty, \tag{14}$$

$$\nabla u_n \to \nabla u \text{ in } L^1(\Omega)^d, \text{ as } n \to \infty, \tag{15}$$

$$\||\nabla u_n|^{q_n(x)}\|_1 \leq C, \text{ for some positive constant } C \text{ not depending on } n. \tag{16}$$

Then $\nabla u \in L^{q(\cdot)}(\Omega)^d$ and

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{q_n(x)}dx \geq \int_\Omega |\nabla u|^{q(x)}dx. \tag{17}$$

3 Main results

In this section we formulate and prove the main result of the paper.
Define the set

$$W_0^{1,p(b(u))}(\Omega) := \left\{ u \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla u|^{p(b(u))} \mathrm{d}x < \infty \right\}.$$

If $1 < p(b(u)) < +\infty$ for all $u \in \mathbb{R}$, this set is a Banach space for norm $\|u\|_{W_0^{1,p(\cdot)}(\Omega)}$, which is equivalent to $\|\nabla u\|_{L^{p(b(u))}(\Omega)}$ in the case of $p(b(u)) \in C(\bar{\Omega})$. If, for some constant $\alpha, p \geq \alpha > 1, p$ and b are continuous, then $W_0^{1,p(b(u))}(\Omega)$ is a closed subspace of $W_0^{1,\alpha}(\Omega)$ then, it is separable and reflexive. In what follows, $W^{-1,\alpha'}(\Omega) = W_0^{1,\alpha}(\Omega)^*$, with $1 < \alpha < +\infty$, denotes as usual the dual space of $W_0^{1,\alpha}(\Omega)$. In the same way we define $W_0^{1,q(b(u))}(\Omega)$.
Before we prove the existence theorem we place some restrictions to the exponents and assume that $p(\cdot)$ and $q(\cdot)$ are real functions satisfying the following:

$$p, q \ \text{ are continuous} \quad \text{and} \quad 1 < \alpha \leq q < p \leq \beta < \infty, \tag{18}$$

for some constants α and β. With respect to constant α, we define domain $W_0^{1,\alpha}(\Omega)$ of the real map $b(\cdot)$, and additionally we impose the following:

$$b \quad \text{is continuous} \ , \quad b \quad \text{is bounded} \tag{19}$$

that is, $b(\cdot)$ sends bounded sets of $W_0^{1,\alpha}(\Omega)$ into bounded sets of $\mathbb{R}$. Now, we give a definition of entropy solutions for the elliptic problem (1).

Definition 1. *A measurable function u with $T_k(u) \in W_0^{1,p(b(u))}(\Omega)$ is said to be an entropy solution for the problem* (1)*, if*

$$\int_\Omega |\nabla u|^{p(b(u))-2} \nabla u \cdot \nabla T_k(u-\varphi) dx + \int_\Omega |\nabla u|^{q(b(u))-2} \nabla u \cdot \nabla T_k(u-\varphi) dx \leqslant \int_\Omega f T_k(u-\varphi) \mathrm{d}x, \tag{20}$$

for all $\varphi \in C_0^1(\Omega)$ and for every $k > 0$.

It is technically useful to extend the above definition of entropy solution to more general truncation functions than T_k. We introduce the class $\mathcal{T}$ of functions $T \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying:

$$T(0) = 0 \text{ and } T(-t) = -T(t), T'(t) \geqslant 0, \text{ for any } t \in \mathbb{R},$$
$$T'(t) = 0 \text{ for any } t \text{ large enough and } T''(t) \leqslant 0, t \geqslant 0.$$

Lemma 2. *The entropy condition* (20) *is equivalent to the following statement that*

$$\int_\Omega |\nabla u|^{p(b(u))-2} \nabla u \cdot \nabla T(u-\varphi) dx + \int_\Omega |\nabla u|^{q(b(u))-2} \nabla u \cdot \nabla T(u-\varphi) dx \leqslant \int_\Omega f T(u-\varphi) dx, \tag{21}$$

for all $\varphi \in C_0^1(\Omega)$ and for every $T \in \mathcal{T}$.

Remark 1. *The proof of Lemma 2 is similar to Lemma* 3.2 *in Benilan et al. (1995) and we will omit it here.*

We remark that quantities $p(b(u))$ and $q(b(u))$ reduce to real numbers and not functions. Consequently, we can treat variable exponent Sobolev spaces in Definition 1 as constant exponent Sobolev spaces.

Theorem 1. *Assume that* (18) *and* (19) *hold together with $f \in L^1(\Omega)$. Then there exists at least one entropy solution of the problem* (1) *in the sense of the Definition 1.*

The proof of Theorem (1) is divided into into several steps.
Step 1: The approximate problem.
We consider the following approximate problem of the problem (1)

$$(\mathcal{P}_n)\begin{cases} -\mathrm{div}(|\nabla u_n|^{p(b(u_n))-2}\nabla u_n) - \mathrm{div}(|\nabla u_n|^{q(b(u_n))-2}\nabla u_n) = f_n \text{ in } \Omega \\ u_n = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases}$$

where f_n is a sequence of $C_0^\infty(\Omega)$ functions strongly converging to f in L^1 such that $\|f_n\|_{L^1} \leq 2\|f\|_{L^1}$.
By employing the arguments in Theorem 2.3.2 of Yanru (2021), we obtain the following result. Then, based on this result, we could get the existence of approximate solutions to (1) with $f \in L^1(\Omega)$.

Theorem 2. *Let $\Omega \subset \mathbb{R}^N, N \geqslant 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$. Assume that* (18) *hold together with*

$$f \in W^{-1,\alpha'}(\Omega).$$

Then the problem $(\mathcal{P}_n)$ admits at least one weak solution $u_n \in W_0^{1,p(u_n)}(\Omega)$ in the following sense

$$\int_\Omega |\nabla u_n|^{p(b(u_n))-2}\nabla u_n \cdot \nabla\varphi dx + \int_\Omega |\nabla u_n|^{q(b(u_n))-2}\nabla u_n \cdot \nabla\varphi dx = \int_\Omega f_n\varphi dx, \tag{22}$$

for all $\varphi \in W_0^{1,p(u_n)}(\Omega) \cap W_0^{1,q(u_n)}(\Omega)$.

Our aim is to prove that a subsequence of these approximate solutions $\{u_n\}$ converges to a measurable function u, which is an entropy solution to (1).
Step 2: a priori estimate.

Proposition 4. *If u is an entropy solution to problem* (1)*, then there exists a positive constant C such that for all $k > 1$*

$$\mathrm{meas}\{|u| > k\} \leqslant \frac{C(A+1)^{\alpha^*/\alpha}}{k^{\alpha^*(1-1/\alpha)}},$$

where α^* is the Sobolev embedding exponent with respect to α.

Proof.
Choosing $\varphi = 0$ as a test function in (20), we obtain

$$\begin{aligned}&\int_\Omega |\nabla T_k(u)|^{p(b(u))}\,\mathrm{d}x + \int_\Omega |\nabla T_k(u)|^{q(b(u))}\,dx \\ &= \int_{\{|u|\leqslant k\}} |\nabla u|^{p(b(u))}\mathrm{d}x + \int_{\{|u|\leqslant k\}} |\nabla u|^{q(b(u))}\mathrm{d}x \leqslant k\|f\|_{L^1(\Omega)},\end{aligned}$$

which implies that for all $k > 1$,

$$\frac{1}{k}\int_\Omega |\nabla T_k(u)|^{p(b(u))}\,\mathrm{d}x \leqslant A := \|f\|_{L^1(\Omega)}, \tag{23}$$

and

$$\frac{1}{k}\int_\Omega |\nabla T_k(u)|^{q(b(u))}\,\mathrm{d}x \leqslant A := \|f\|_{L^1(\Omega)}.$$

Since

$$W_0^{1,p(b(u))}(\Omega) \hookrightarrow W_0^{1,\alpha}(\Omega) \hookrightarrow L^{\alpha^*}(\Omega).$$

Then for every $k > 1$

$$\begin{aligned}\|T_k(u)\|_{L^{\alpha^*}(\Omega)} &\leqslant C\,\|\nabla T_k(u)\|_{L^{p(b(u))}(\Omega)}\\ &\leqslant C\left(\int_\Omega |\nabla T_k(u)|^{p(b(u))}\,\mathrm{d}x\right)^{\delta} \leqslant C(Ak)^{\delta},\end{aligned}$$

where

$$\delta = \begin{cases} \frac{1}{\alpha} & \text{if } \|\nabla T_k(u)\|_{L^{p(b(u))}(\Omega)} \geqslant 1 \\ \frac{1}{\beta} & \text{if } \|\nabla T_k(u)\|_{L^{p(b(u))}(\Omega)} \leqslant 1. \end{cases}$$

Noting that $\{|u| \geqslant k\} = \{|T_k(u)| \geqslant k\}$, we have

$$\operatorname{meas}\{|u| > k\} \leqslant \left(\frac{\|T_k(u)\|_{L^{\alpha^*}(\Omega)}}{k}\right)^{\alpha^*} \leqslant \frac{CA^{\delta\alpha^*}}{k^{\alpha^*(1-\delta)}} \leqslant \frac{C(A+1)^{\alpha^*/\alpha}}{k^{\alpha^*(1-1/\alpha)}}.$$

This completes the proof.

Step 3: The convergence in measure of $\{u_n\}$.

For every $\epsilon > 0$ and every positive integer k, we have

$$meas\left\{|u_n - u_m| > \epsilon\right\} \leq meas\left\{|u_n| > k\right\} + meas\left\{|u_m| > k\right\} + meas\left\{|T_k\left(u_n\right) - T_k\left(u_m\right)| > \epsilon\right\}.$$

Choosing $T_k\left(u_n\right)$ as a test function in (22), we get

$$\int_\Omega \left|\nabla T_k\left(u_n\right)\right|^{p(b(u_n))} dx + \int_\Omega \left|\nabla T_k\left(u_n\right)\right|^{q(b(u_n))} dx \leqslant k\left\|f_n\right\|_{L^1(\Omega)} \leqslant 2k\|f\|_{L^1(\Omega)}. \tag{24}$$

By Hölder's inequality and (24), we have

$$\begin{aligned}\int_\Omega \left|\nabla T_k\left(u_n\right)\right|^{\alpha} dx &\leqslant C\left(\int_\Omega \left|\nabla T_k\left(u_n\right)\right|^{p(b(u_n))} dx\right)^{\frac{\alpha}{p(b(u_n))}}\\ &\leqslant C\left(\int_\Omega \left|\nabla T_k\left(u_n\right)\right|^{p(b(u_n))} dx + 1\right) \leqslant C.\end{aligned} \tag{25}$$

We deduce that $\{T_k\left(u_n\right)\}$ is convergent in $L^q(\Omega)$ with $q \in [1, \alpha^*)$. It follows from Proposition 4 that

$$\limsup_{n,m\to\infty} \operatorname{meas}\left\{|u_n - u_m| > \epsilon\right\} \leqslant C\left(\|f\|_{L^1(\Omega)}\right) k^{-\tilde{\alpha}},$$

where $\tilde{\alpha} = \alpha^*(1 - 1/\alpha) > 0$.

Because k is arbitrary, we prove that

$$\limsup_{n,m\to\infty} \operatorname{meas}\left\{|u_n - u_m| > \epsilon\right\} = 0,$$

which implies the convergence in measure of $\{u_n\}$. Then there exists a subsequence (still denoted by u_n) in Ω such that

$$u_n \to u \quad \text{a.e in} \quad \Omega. \tag{26}$$

Step 4: The convergence almost everywhere in Ω of $\{\nabla u_n\}$.

We first prove that $\{\nabla u_n\}$ is a Cauchy sequence in measure. Let $\delta > 0$, and set

$$\begin{aligned}E_1 &:= \left\{x \in \Omega : |\nabla u_n| > h\right\} \cup \left\{x \in \Omega : |\nabla u_m| > h\right\},\\ E_2 &:= \left\{x \in \Omega : |u_n - u_m| > 1\right\}\end{aligned}$$

and

$$E_3 := \left\{x \in \Omega : |\nabla u_n| \leqslant h, |\nabla u_m| \leqslant h, |u_n - u_m| \leqslant 1, |\nabla u_n - \nabla u_m| > \delta\right\},$$

where h will be chosen later. Obviously we have

$$\{x \in \Omega : |\nabla u_n - \nabla u_m| > \delta\} \subset E_1 \cup E_2 \cup E_3.$$

We may draw a subsequence still denoted by the original sequence such that

$$\nabla T_k(u_n) \to \eta_k \quad \text{in } \left(L^{\alpha}(\Omega)\right)^N.$$

From (26), we deduce that $\eta_k = \nabla T_k(u)$ a.e. in Ω. Moreover, from Lemma 1 we know that $\nabla T_k(u) \in \left(L^{p(\cdot)}(\Omega)\right)^N$ and

$$\liminf_{n\to\infty} \int_\Omega |\nabla T_k(u_n)|^{p(b(u_n))} \,\mathrm{d}x \geqslant \int_\Omega |\nabla T_k(u)|^{p(b(u))} \,dx.$$

For $k > 0$, we have

$$\{x \in \Omega : |\nabla u_n| \geqslant h\} \subset \{x \in \Omega : |u_n| \geqslant k\} \cup \{x \in \Omega : |\nabla T_k(u_n)| \geqslant h\}.$$

Thus, from (25) and Proposition 4, there exist constants $C > 0$ such that

$$\operatorname{meas}\{x \in \Omega : |\nabla u_n| \geqslant h\} \leqslant \frac{C}{k^{\alpha^*(1-1/\alpha)}} + \frac{C}{h^{\alpha}}.$$

By choosing $k = Ch^{\alpha/(\alpha^*(1-1/\alpha))}$, we deduce that

$$\operatorname{meas}\{x \in \Omega : |\nabla u_n| \geqslant h\} \leqslant \frac{C}{h^{\alpha}}.$$

Let $\varepsilon > 0$. We may choose $h = h(\varepsilon)$ large enough such that

$$\operatorname{meas}(E_1) \leqslant \frac{\varepsilon}{3}, \quad \text{for all } n, m \geqslant 0. \tag{27}$$

On the other hand, since $\{u_n\}$ is a Cauchy sequence in measure. Then there exists $N_1(\varepsilon) \in \mathbb{N}$ such that

$$\operatorname{meas}(E_2) \leqslant \frac{\varepsilon}{3}, \qquad \text{for all } n, m \geqslant N_1(\varepsilon). \tag{28}$$

Notice that, for all $q > 1$ and for all $\xi, \zeta \in \mathbb{R}^N$ with $|\xi|, |\zeta| \leqslant h, |\xi - \zeta| \geqslant \delta$, there exists a real valued function $m(h, \delta) > 0$ such that

$$\left(|\xi|^{q-2}\xi - |\zeta|^{q-2}\zeta\right) \cdot (\xi - \zeta) \geqslant m(h, \delta) > 0.$$

By taking $T_1(u_n - u_m)$ as a test function in the approximation equation (22) and integrating on E_3, we get

$$\begin{aligned}
& m(h,\delta)\operatorname{meas}(E_3) \\
& \leqslant \int_{E_3}\left[|\nabla u_n|^{p(b(u_n))-2}\nabla u_n - |\nabla u_m|^{p(b(u_n))-2}\nabla u_m\right]\cdot(\nabla u_n-\nabla u_m)\,dx \\
& \qquad + \int_{E_3}\left[|\nabla u_n|^{q(b(u_n))-2}\nabla u_n - |\nabla u_m|^{q(b(u_n))-2}\nabla u_m\right]\cdot(\nabla u_n-\nabla u_m)\,dx \\
& = \int_{E_3}\left[|\nabla u_m|^{p(b(u_m))-2}\nabla u_m - |\nabla u_m|^{p(b(u_n))-2}\nabla u_m\right]\cdot(\nabla u_n-\nabla u_m)\,dx \\
& \qquad + \int_{E_3}\left[|\nabla u_m|^{q(b(u_m))-2}\nabla u_m - |\nabla u_m|^{q(b(u_n))-2}\nabla u_m\right]\cdot(\nabla u_n-\nabla u_m)\,dx \\
& \qquad\qquad + \int_{E_3}[f_n-f_m]\,T_1(u_n-u_m)\,dx \\
& \leqslant \int_{E_3}|\nabla u_m|^{\eta-1}\,|\log|\nabla u_m||\cdot|\nabla u_n-\nabla u_m|\cdot|p(b(u_m))-p(b(u_n))|\,dx \\
& \qquad + \int_{E_3}|\nabla u_m|^{\rho-1}\,|\log|\nabla u_m||\cdot|\nabla u_n-\nabla u_m|\cdot|q(b(u_m))-q(b(u_n))|\,dx \\
& \qquad\qquad + \|f_n-f_m\|_{L^1(\Omega)} \\
& \leqslant 2h^\beta\log h\cdot\int_\Omega\left(|p(b(u_m))-p(b(u_n))|+|q(b(u_m))-q(b(u_n))|\right)dx+\|f_n-f_m\|_{L^1(\Omega)} := \alpha_{n,m}.
\end{aligned}$$

Here we used the facts that $h \gg 1$, relation (18), the definition of E_3 and the mean value theorem with η and ρ taking values between $p(b(u_m))$ and $p(b(u_n))$ and between $q(b(u_m))$ and $q(b(u_n))$ respectively, in the last two inequalities. By using the Lebesgue dominated convergence theorem we obtain

$$\operatorname{meas}(E_3) \leqslant \frac{\alpha_{n,m}}{m(h,\delta)} \leqslant \frac{\varepsilon}{3},$$

for all $n,m \geqslant N_2(\varepsilon,\delta)$. Combining the estimates above we get

$$\operatorname{meas}\ \{x\in\Omega : |\nabla u_n-\nabla u_m|>\delta\} \leqslant \varepsilon, \qquad \text{for all } n,m \geqslant \max\{N_1,N_2\},$$

hence $\{\nabla u_n\}$ is a Cauchy sequence in measure. Then we can choose a subsequence (denote it by the original sequence) such that

$$\nabla u_n \to v \quad \text{ a.e. in } \Omega.$$

Thus, using Proposition 3 and the fact that $\nabla T_k(u_n) \to \nabla T_k(u)$ in $(L^\alpha(\Omega))^N$, we deduce that v coincides with the very weak gradient of u almost everywhere. Therefore, we have

$$\nabla u_n \to \nabla u \quad \text{ a.e. in } \Omega. \tag{29}$$

Step 5: Passing to the limit.
In order to prove (21) we take $T \in \mathcal{T}$ bounded by $s_0 > 0$ such that $T'(s) = 0$, for any $s \geqslant s_0$. Now we choose $T(u_n-\phi)$ as a test function in (22) for $\phi \in C_0^1(\Omega)$. Then

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p(b(u_n))-2}\nabla u_n\cdot\nabla T(u_n-\phi)\,\mathrm{d}x \\
&\quad + \int_\Omega |\nabla u_n|^{q(b(u_n))-2}\nabla u_n\cdot\nabla T(u_n-\phi)\,\mathrm{d}x = \int_\Omega f_n T(u_n-\phi)\,\mathrm{d}x.
\end{aligned}\tag{30}$$

For the first term in the left-hand side of (30), we have

$$\begin{aligned}
\int_\Omega |\nabla u_n|^{p(b(u_n))-2}\nabla u_n\cdot\nabla T(u_n-\phi)\,dx &= \int_\Omega |\nabla u_n|^{p(b(u_n))}\,T'(u_n-\phi)\,dx \\
&\quad - \int_\Omega |\nabla u_n|^{p(b(u_n))-2}\nabla u_n T'(u_n-\phi)\cdot\nabla\phi dx.
\end{aligned}\tag{31}$$

From (26), (29) and Fatou's Lemma, we deduce

$$\int_\Omega |\nabla u|^{p(b(u))} T'(u-\phi) \mathrm{d}x \leqslant \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p(b(u_n))} T'(u_n-\phi)\, dx. \tag{32}$$

We now focus our attention on the second term in the right-hand side of (31).
We note that, if $L = s_0 + \|\phi\|_{L^\infty(\Omega)}$

$$\left| |\nabla u_n|^{p(b(u_n))-2} \nabla u_n T'(u_n-\phi) \right| \leqslant C \left|\nabla T_L(u_n)\right|^{p(b(u_n))-1}.$$

Using (24), we have $\left\{|\nabla u_n|^{p(b(u_n))-2} \nabla u_n T'(u_n-\phi)\right\}$ is bounded in $\left(L^{p'(b(u_n))}(\Omega)\right)^N \subset \left(L^{\beta'}(\Omega)\right)^N$.
Since $u_n \to u$ a.e. in Ω and $\nabla u_n \to \nabla u$ a.e. in Ω, we have

$$|\nabla u_n|^{p(b(u_n))-2} \nabla u_n T'(u_n-\phi) \to |\nabla u|^{p(b(u))-2} \nabla u T'(u-\phi) \quad \text{a.e. in } \Omega,$$

which implies that

$$|\nabla u_n|^{p(b(u_n))-2} \nabla u_n T'(u_n-\phi) \rightharpoonup |\nabla u|^{p(b(u))-2} \nabla u T'(u-\phi) \quad \text{in } \left(L^{\beta'}(\Omega)\right)^N.$$

As $\phi \in C_0^1(\Omega)$, we get

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p(b(u_n))-2} \nabla u_n T'(u_n-\phi) \cdot \nabla \phi dx \\ &\quad \longrightarrow \int_\Omega |\nabla u|^{p(b(u))-2} \nabla u T'(u-\phi) \cdot \nabla \phi dx, \quad \text{as } n \to \infty. \end{aligned} \tag{33}$$

Combining (31), (32) and (33), we deduce

$$\int_\Omega |\nabla u|^{p(b(u))-2} \nabla u \nabla T(u-\phi) dx \leqslant \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p(b(u_n))-2} \nabla u_n \nabla T(u_n-\phi)\, dx. \tag{34}$$

In the same way we show that

$$\int_\Omega |\nabla u|^{q(b(u))-2} \nabla u \nabla T(u-\phi) dx \leqslant \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{q(b(u_n))-2} \nabla u_n \nabla T(u_n-\phi)\, dx. \tag{35}$$

Now, we consider the right hand side of (30), since $f_n \to f$ in $L^1(\Omega)$ then

$$\lim_{n\to\infty} \int_\Omega f_n T(u_n-\phi)\, dx = \int_\Omega f T(u-\phi)\, dx. \tag{36}$$

Using (30), (34), (35) and (36) we get

$$\begin{aligned} \int_\Omega |\nabla u|^{p(b(u))-2} \nabla u \cdot \nabla T(u-\phi)\, dx + \int_\Omega |\nabla u|^{q(b(u))-2} \nabla u \cdot \nabla T(u-\phi)\, dx \\ \leqslant \int_\Omega f T(u-\phi)\, dx, \end{aligned} \tag{37}$$

for $T \in \mathcal{T}$ and $\phi \in C_0^1(\Omega)$. Therefore, from Lemma 2 we complete the proof of the existence of entropy solutions.

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